Final Report

Junior Trimester Program
Kinetic Theory

This is the final report for the group *LFD and Inelastic Boltzmann equation* at the Junior Trimester Program (JTP) at the Hausdorff Institute for Mathematics (HIM) in the spring of 2019. The members of our group were Ricardo J. Alonso (Jun. 30 – Jul. 17), Véronique Bagland (Jul. 01 – Jul. 13) and Bertrand Lods (Jul. 01 – Jul. 06).

Scientific activities

The group acknowledges the excellent working conditions provided by the Hausdorff Research Institute for Mathematics (HIM). We are particularly grateful for the funding provided for the stay and the facilities offered by HIM. We feel very privileged to have had this opportunity to work, even for a short period, at the HIM. The whole JTP has been a very productive, and pleasant experience and we enjoyed being in touch during our stay with the other group members ~~during our stay~~. During our period of stay, no workshop were organised at the HIM but we attended various of the *Trimester Seminars*. A member of the group (B. Lods) presented some of the previous results of the group in a talk given within the *Seminar Series Coagulation-fragmentation*.

Research and output

During our stay at the HIM we discussed and worked on the following questions related to the kinetic description of *quantum plasmas and gases* as well as *granular gases*:

(1) ***Long-time behavior of Landau-Fermi-Dirac equation.*** As planned in the research project, we have been able to make progress on the study of Landau-Fermi-Dirac equation. Such a model is a modification of the Landau equation for collisional plasma to quantum particles taking into account the Pauli exclusion principle. The three group members finalise at the HIM a first work regarding the long-time behaviour of the solution to the spatially homogeneous equation for interactions corresponding to the so-called *hard potentials*. The results obtained in this paper has been published in a high-quality international journal [1]. We exhibit in this work an exponential convergence towards to equilibrium by exploiting the appearance and uniform boundedness of moments and regularity norms for the solutions to Landau-Fermi-Dirac equation. The rate of convergence is obtained by combining a careful spectral analysis (for close-to-equilibrium solutions) with a thorough analysis of the entropy production properties of the collision operator. Such an approach is robust enough to be applied successfully to the more delicate case of *soft potentials* interactions for which the convergence rate is not expected to be exponential anymore. We started, during our stay at HIM, to work on a general study of the Landau-Fermi-Dirac equation for soft potential interactions. A few months after the JTP, we had the opportunity

to get the help of Laurent Desvillettes on this research topics and the collaboration results in one submitted paper [2] and a paper which is at its final stage [3] before submission. In the first of this paper, we give a complete and unified study of the entropy production for the Landau-Fermi-Dirac equation (with hard or moderately soft potentials) in terms of a weighted relative Fisher information adapted to this equation. Such estimates are used for studying the large time behaviour of the equation, as well as for providing new a priori estimates (in the soft potential case). In the second paper, we establish a nearly optimal algebraic relaxation to equilibrium for solutions to the LFD equation with moderately soft potentials. The analysis is highly non trivial and combines uniform in time estimates for statistical moments, L^p-norm generation and Sobolev regularity together with an innovative level set analysis in the spirit of De Giorgi.

(2) ***Entropy production for Quantum Boltzmann equation.*** In the spirit of the previous point, we also started during our stay at HIM, a comprehensive analysis of the entropy production for the Quantum Boltzmann equation, for both Fermi-Dirac and Bose-Einstein description. The geometry of the collision here makes the analysis significantly more technical than the one of the related Landau equation. We obtained already encouraging results at the HIM but such partial results still need to be finalised and complemented with a careful study of the solutions to the Quantum Boltzmann equation. Indeed, several of the entropy production estimates we obtained so far are strongly related to some *a priori* regularity estimates. To apply the newly obtained tools to the study of the long-time behaviour of the associated Quantum Boltzmann equation, we need to prove that solutions to such equations do indeed satisfy those regularity estimates. This is a quite delicate problem which is, for the members of the group, a challenging research project on a mid-term basis.

(3) ***Regularity theory and uniqueness of steady states for 1D inelastic Boltzmann.*** In collaboration with José A. Cañizo and Sebastian Throm (member of the group lead by J. A. Cañizo) we started to work at HIM on the propagation of regularity and uniqueness of the steady state for a one dimensional Boltzmann equation with dissipative interactions on which our group worked in recent years. These two questions are well-understood for Boltzmann equation with dissipative interactions in higher dimension, but their analysis for the one-dimensional version of the equation is quite delicate. For the propagation of regularity, the main obstacle is the lack of particle scattering. Indeed, mathematically speaking, particle scattering represent some kind of diffusion process that help to find coercivity estimates. Contrary to particle systems in two or more dimensions, this is not the case in 1D inelastic Boltzmann where such coercivity is very weak. As far as the uniqueness of the steady state is concerned, the question is answered in higher dimension by some perturbative argument on the inelasticity parameter and exploiting the results known for the classical Boltzmann equation (with conservative interactions). Such an approach does not apply to the 1D inelastic Boltzmann equation since its elastic counterpart

is meaningless. We discussed in HIM the possibility of tackling both these questions using perturbative techniques not related to the inelasticity parameter but with the strength of the potential interactions. Indeed, in the case of constant interaction potentials (pseudo-Maxwellian interactions), the uniqueness and propagation of regularity are known to hold thanks to a clever use of Fourier analysis and Fourier-based distances. We believe it should be possible to exploit these known results as a pivot for a perturbative approach when dealing with hard potential interactions whose strength γ is small enough (the pseudo-Maxwellian case corresponding to $\gamma = 0$). This approach is completely new for the study of the Boltzmann equation even it is reminiscent to similar questions addressed in the study of coagulation-fragmentation processes. Before attacking this perturbative approach, we started to strengthen the results obtained in the pseudo-maxwellian and reformulate them in some more robust Banach space framework (not depending anymore on Fourier-based metrics which are ineffective for hard potentials). The results obtained so far are already satisfactory but we aim to complement them with the aforementioned implementation of the perturbative approach which is much more delicate. This project is still an active field of research for our group even if the progresses made in the previous two projects delayed us a bit in this specific one.

(4) ***Contractivity for Smoluchowski's coagulation equation*** Inspired by the discussions on the Fourier-based metrics for the 1D-Boltzmann equation and the analogy with models for coagulation-fragmentation, we also discussed at the HIM the adaptation of these metrics to the study of Smoluchowski's equation. These discussions result in the publication of the paper [4] which revisit the study of the so-called solvable kernels for Smoluchowski's equation introducing for this purpose metrics based upon the Laplace transform technique instead of Fourier transform. Together with José A. Cañizo and Sebastian Throm, research is still active in this direction, in particular trying to adapt the techniques introduced in [4] to some other kernels for which uniqueness of the steady state is known to hold (inverse power law kernel).

References

[1] R. J. Alonso, V. Bagland, & B. Lods, Long time dynamics for the Landau-Fermi-Dirac equation with hard potentials, *J. Differential Equations* **270** (2021), 596–663.

[2] R. J. Alonso, V. Bagland, L. Desvillettes, & B. Lods, About the use of Entropy dissipation for the Landau-Fermi-Dirac equation, submitted for publication, 2020.

[3] R. J. Alonso, V. Bagland, L. Desvillettes, & B. Lods, About the Landau-Fermi-Dirac equation with moderately soft potentials, work in progress, 2021.

[4] J. Canizo, B. Lods, & S. Throm, Contractivity for Smoluchowski's coagulation equation with solvable kernels, *Bulletin of the London Mathematical Society*, to appear, 2020.

Final Report of the Group

Scaling limits for particle systems

Junior Hausdorff Trimester Program "Kinetic Theory"

Period of stay: May 19 - June 30, 2019

The *Scaling limits for particle systems* group participated in the first half of the Junior Trimester Program at the Hausdorff Institute for Mathematics in Bonn, during the summer of 2019. The members of the group were Roberta Bianchini (IAC CNR, LJLL Paris at the time), Lingbing He (Tsinghua University), Chiara Saffirio (University of Basel, University of Zurich at the time) and Sergio Simonella (ENS Lyon). One of the intended group members, Serena Cenatiempo (GSSI), was not able to attend. She however had a key role in the workshop organised by the group, both from the practical (such as dealing with emails to speakers and participants) and scientific viewpoint (such as by contributing in the choice of speakers).
All the members of the group share the opinion that their stay at HIM was very fruitful and pleasant. They enjoyed the scientific interactions among them, with members of other groups and with visitors who were attending the several workshops that have been organised. Some of the ideas arisen and discussed, and the collaborations started during the stay at HIM will lead the research of the group for the next years. The members of the group are therefore thankful to the HIM for giving them such an opportunity.

1 Overview on topics and goals

A major question in kinetic theory concerns the rigorous derivation of effective macroscopic equations from the microscopic laws of classical and quantum mechanics. The prototype of kinetic equations has been introduced by Maxwell and Boltzmann at the end of the XIX century while attempting at a realistic description of the time evolution of a rarefied gas at a kinetic level, namely on quantities which averages are susceptible of measurement. In recent years, the problem of rigorously deriving effective evolution equations from many-particle classical or quantum systems has seen an increasing interest from the mathematic community and several progresses have been done. However, some central questions are still open. The group *Scaling limits for particle systems* focused on some of those during the stay at HIM.

More precisely: the effect of long range interactions in the Boltzmann description and the related Landau equation (Cf. 1, 2 and 4, and **a**, **b**, **d**, **f** in Section 3); the role and characterisation of pathological configurations in the derivation of the Boltzmann equation (Cf. 2, 3 and **e** in Section 3); the hydrodynamics of the Boltzmann equation (Cf. **c** in Section 3).

2 Activities

Organisation of the workshop "Effective equations: frontiers in classical and quantum systems" (June 24-28, 2019).
The workshop was a great opportunity to enhance interactions with the most prominent experts in the field of derivation of effective equations. The members of the group very much profited from the talks and the discussions during the breaks. Some collaborations started thanks to these discussions (Cf. items **d** and **e** in Section 3). Moreover, the workshop was largely attended and the feedbacks of the participants were extremely positive.

Invitation of Prof. Bobylev and Prof. Pulvirenti.
The group benefited from interesting and fruitful discussions with Prof. Bobylev (Keldysh Institute of Applied Mathematics in Moscow) and Prof. Pulvirenti (University of Rome La Sapienza). The two invited professors also contributed two informal lectures, two official talks and one mini-course during their stay at HIM.

Informal seminars and discussions.
The group planned several discussion sessions and informal seminars among the members of the group and in collaboration with other groups (in particular with A. Nota and R. Winter at the University of Bonn). Among them, the seminar by one of the member of the group (S. Simonella) on the derivation of the Boltzmann equation with short range interaction potentials and several blackboard discussions leaded by L. He on the quantum Boltzmann equation and its relation to the Landau equation.

Active participation to workshops.
The group actively participated in the summer school *Trails in Kinetic Theory: fundamental aspects and numerical methods* (May 20-24,2019) and the workshops *Qualitative behaviour of Kinetic Equations: numerical and theoretical aspects* (June 3-7, 2019), *Analytical and computational problems for mixtures and plasma dynamics* (June 17-21, 2019) organised by other participants in the Junior Trimester Program.

3 Scientific outputs

The outcomes of the stay at HIM are summarised in this section.

<u>Scientific publications and preprints</u>

1. L. He, Y. Zhou. *Boltzmann equation with cutoff Rutherford scattering cross section near Maxwellian.* arXiv:2009.07598

2. M. Pulvirenti, S. Simonella. *A brief introduction to the scaling limits and effective equations in kinetic theory.* In: Trails in Kinetic Theory, SEMA SIMAI Springer Series 25, ed. Albi Merino-Aceituno Nota Zanella (2021)

3. M. Pulvirenti and S. Simonella. On the cardinality of collisional clusters for hard spheres at low density. Disc.&Cont.Dyn.Syst. (in press)

4. A. Nota, C. Saffirio, S. Simonella. *Two-dimensional Lorentz process for magnetotransport: Boltzmann-Grad limit.* arXiv:1910.12983

<u>New projects and collaborations</u>

a. Extension of the Landau's derivation to the potentials with inverse power law: work in progress (L. He, J.C. Jiang and Y. Zhou).

b. Grazing collision limit towards the Landau equation (L. He, M. Pulvirenti, S. Simonella, R. Winter).

c. A priori estimate on the Boltzmann equations, which should provide interesting informations at the hydrodynamic scale (R. Bianchini, C. Saffirio).

d. A Kac model for the Landau equation (C. Saffirio, R. Strain, R. Winter).

e. Triple collisions in the Boltzmann picture (I. Gamba, N. Pavlovic, C. Saffirio).

f. The Boltzmann-Vlasov equation: work in progress (L. Desvillettes, C. Saffirio, S. Simonella).

The members of the group very much profited from constant discussions with Prof. Spohn (TU Munich) and Prof. Velázquez (University of Bonn).

Report of

Group Report

Junior Hausdorff Trimester Program

Kinetic Theory

Phase transitions, long-time behaviour and variational formulation of kinetic equations

and

Kinetic Theory of Growth Processes

Group members

Junior members: Marco Bonacini; Emre Esenturk; Antonio Esposito; Rishabh Gvalani; André Schlichting; Markus Schmidtchen
Senior members: Philippe Laurencot; Robert Pego, his PhD student Truong-Son Van

Personal feedback (selection)

Antonio Esposito: Overall, my stay in Bonn and at HIM was fantastic. The organisation of the program was perfect as well as working conditions. The offices and the equipment were adequate. The stuff of HIM was always ready to provide us support, even regarding our accommodation in Bonn. The success of program was possible thanks to the unique atmosphere of HIM.

Philippe Laurencot: The working conditions were almost perfect with a comfortable and well-located office, nearby printers, generous supply of coffee and cakes, and the administrative staff prior to my coming and during my stay was very efficient. The only drawback was the Internet access which was fluctuating in the building where my office was, whatever access I used, eduroam or the one provided by the center. As for the housing, it was simply great, well furnished and well located. I really enjoyed my stay in Bonn and would be eager to be invited again!

Robert Pego: The housing I had was excellent, and in a very convenient location. The design and environment of the Institute was very good indeed for encouraging discussion and collaboration. I recall only a couple of little things that caused any difficulties– heat on the top floor during the hottest days in June and July, and a weak wireless signal in the top floor office where my desk was.

Markus Schmidtchen: Staying at the HIM was truly rewarding. The atmosphere was collegial and sociable. In summary, the workshop was extremely enjoyable and worthwhile.

Organized events

- Our group was part in organizing the workshop *Probabilistic and variational methods in kinetic theory*, which had in the very first week of the program the role of a kickoff event for the trimester. We as a group got many inputs, ideas and inspirations from the talks for the following research.
- Our group organized the lecture series by Martin Evans and Stefan Grosskinsky on stochastic models for condensation phenomena.
- Our group co-organized and actively participated in the Trimester seminar series on *coagulation-fragmentation*.

Research output

- **Marco Bonacini**, **Barbara Niethammer** and **Juan Velazquez** investigated in [BNV19a, BNV19b] the stability of a special class of solutions to a coagulation-fragmentation equation with coagulation kernels close to the diagonal kernel and diagonal fragmentation kernels.
- Marco Di Francesco, **Antonio Esposito**, and **Markus Schmidtchen** obtained in [DFES20] the many-particle limit for a system of interaction equations driven by Newtonian potentials.
- Constantin Eichenberg and **André Schlichting** proved in [ES21] rigorously the coarsening behavior towards the self-similar profile for the exchange-driven growth process with product kernels.
- **Antonio Esposito**, Francesco S. Patacchini, **André Schlichting**, and Dejan Slepčev introduced in [EPSS21] a variational description of aggregation equations on graphs and general continuous nonlocal state spaces.
- **Emre Esenturk** and **Juan Velazquez** proved in [EV21] the longtime behavior of the exchange-driven growth model for kernels satisfying either a detailed balance condition or specific monotonicity assumptions.
- **Rishabh Gvalani** and **André Schlichting** obtained in [GS20] a mountain pass theorem in the space of probability measures with applications to metastability of stochastic interacting particle systems.
- **Philippe Laurencot** obtained in [Lau20] inspired by discussions with the participants Marina Ferreira and Juan J.L. Velazquez a characterization of stationary solutions to Smoluchowski's coagulation equation with sources.
- Hiroyoshi Mitake, Hung V. Tran and **Truong-Son Van** connected in [TV20, MTV20] a critical coagulation-fragmentation equation with multiplicative coagulation kernel and constant fragmentation kernel to the study of viscosity solutions to a new singular Hamilton-Jacobi equation. They proved wellposedness, regularity and long-time behaviors of viscosity solutions to the Hamilton-Jacobi equation in certain regimes, which have implications to wellposedness and long-time behaviors of *mass-conserving* solutions to the coagulation-fragmentation equation.

Collaborations and works in progress

B. Niethammer, R. Pego, A. Schlichting, and J. Velazquez investigated a bubbleator model, which consists of a cluster growth model with input of small droplets and decay of large clusters. They obtained a formal limit model of the form of a hyperbolic equation together with a nonlocal boundary condition at the origin and proved rigorously the occurence of a supercritical Hopf bifurcation for this model.

A. Esposito, R. Gvalani, A. Schlichting, and M. Schmidtchen worked during the time of the trimester on the variational description of the aggregation and homogeneous inelastic Boltzmann equation. They found a promising and novel nonlocal gradient structures providing a variational description for both equations as gradient flows of the kinetic energy.

Ph. Laurencot started several works still in progress dealing in spirit of [Lau20] with stationary solutions to coagulation-fragmentation equations for general coefficients, self-similar solutions for coagulation-fragmentation equations with specific choices of coagulation and fragmentation coefficients, and the non-existence issue for the exchange driven model.

M. Ferreira, R. Pego, J. Velazquez initiated a joint project on coagulation-fragmentation dynamics with sources, which will be pursued further when time and funding permit.

References

[BNV19a] Marco Bonacini, Barbara Niethammer, and Juan Velázquez. Solutions with peaks for a coagulation-fragmentation equation. Part I: stability of the tails. *arXiv:1906.08965*, June 2019.

[BNV19b] Marco Bonacini, Barbara Niethammer, and Juan Velázquez. Solutions with peaks for a coagulation-fragmentation equation. Part II: aggregation in peaks. *arXiv:1906.08966*, June 2019.

[DFES20] M. Di Francesco, A. Esposito, and M. Schmidtchen. Many-particle limit for a system of interaction equations driven by newtonian potentials. *preprint arXiv:2008.11106*, 2020.

[EPSS21] Antonio Esposito, Francesco Patacchini, André Schlichting, and Dejan Slepčev. The nonlocal interaction equation on graphs as gradient flow with respect to the upwind transportation metric. *submitted to Arch. Ration. Mech. Anal.*, 12 2021.

[ES21] Constantin Eichenberg and André Schlichting. Self-similar behavior of the exchange-driven growth model with product kernel. *accepted at Commun. Partial. Differ. Equ.*, 2021.

[EV21] Emre Esentürk and Juan Velazquez. Large time behavior of exchange-driven growth, 2021.

[GS20] Rishabh S. Gvalani and André Schlichting. Barriers of the McKean-Vlasov energy via a mountain pass theorem in the space of probability measures. *J. Funct. Anal.*, 279(11):108720, 34, 2020.

[Lau20] Philippe Laurençot. Stationary solutions to smoluchowski's coagulation equation with source. *North-W. Eur. J. of Math.*, 6, 2020.

[MTV20] Hiroyoshi Mitake, Hung V. Tran, and Truong-Son Van. Large time behavior for a Hamilton-Jacobi equation in a critical Coagulation-Fragmentation model. *accepted to Communications in Mathematical Sciences, arXiv:2004.13619 [math]*, September 2020. arXiv: 2004.13619.

[TV20] Hung V. Tran and Truong-Son Van. Coagulation-Fragmentation equations with multiplicative coagulation kernel and constant fragmentation kernel. *accepted to Communications on Pure and Applied Mathematics, arXiv:1910.13424*, July 2020.

FINAL REPORT
JUNIOR TRIMESTER PROGRAM KINETIC THEORY, HIM BONN

ESTHER S. DAUS

- **Group name:**
  Stochastic-PDE Bridge
- **Group members:**
  Julio Backhoff, Esther S. Daus, Helge Dietert, Giacomo Di Gesu, Amit Einav, Josephine Evans, Beatrice Signorello, Bao Q. Tang, Tobias Wöhrer, Junjian Yang.
- **Research interests:**
  Diffusion equations in kinetic theory: degenerate diffusion, hypocoercivity, cross-diffusion.
  Links between ideas from stochastic analysis and kinetic theory.

The group acknowledges the excellent working conditions provided by the Hausdorff Research Institute for Mathematics (HIM) and would also like to thank the staff at the HIM for their great help and support. Their excellent handling of the administrative tasks contributed significantly to the research productivity of the group. The group also highly appreciated the daily tea and cake breaks, which could be used to discuss mathematics and to socialize with other participants of the program. The outstanding funding and working environment for visitors afforded by the trimester program greatly helped our collaborations.

SCIENTIFIC ACTIVITIES

We are particularly grateful for the funding provided to organize the workshop *Probabilistic and Variational Methods in Kinetic Theory*, May 13 - 17, 2019, organized by Esther S. Daus, Giacomo Di Gesu, André Schlichting.[1].

- **Workshop name:** Probabilistic and Variational Methods in Kinetic Theory.
- **Organizers:** Esther S. Daus, Giacomo Di Gesu, André Schlichting.
- **Speakers (25):** Anton Arnold, Giada Basile, Martin Burger, Li Chen, Matias Delgadino, Laurent Desvillettes, Amit Einav, Matthias Erbar, Antonio Esposito, Jo Evans, Simone Fagioli, Joaquin Fontbona, Giambattista Giacomin, Rishabh Gvalani, Franca Hoffmann, Ansgar Jüngel, Shi Jin, Jean-Christophe Mourrat, Francesco Patacchini, Mario Pulvirenti, Angela Stevens, Bao Q. Tang, Hagop Tossounian, Oliver Tse, Boguslaw Zegarlinski.
- **Aim of the workshop:** The workshop aimed to bring the communities from the field of stochastic processes and partial differential equations related to kinetic models together. These include for instance the dynamic behavior of kinetic Fokker-Planck and Vlasov-McKean equation with linear or nonlinear (cross-)diffusion. The workshop aimed to highlight recent advances on hypocoercivity, the rate of convergence, self-similarity, and phase transitions. For this, ideas from the side of stochastic analysis, as well as the variational formulation of gradient flows yielded exciting results in kinetic theory. The workshop aimed to deepen these links and build new bridges. Many of the talks were recorded by the HIM and are available online.

The workshop organized by our group during this trimester was very instructive, as it allowed for a great group of researchers from different areas but common interests to meet and exchange perspectives. It was interesting to see how people from Analysis and Geometry were interested in similar questions as people in the field of Applied Probability, and how they deployed completely different tools to tackle related questions.

LIST OF SOME MATHEMATICAL DISCUSSIONS DURING THE TRIMESTER PROGRAM

Below we give an overview of the progress made by the researchers in our group that was facilitated by the HIM trimester program.

[1] https://www.him.uni-bonn.de/programs/past-programs/past-junior-trimester-programs/kinetic-theory-2019/workshop-probabilistic-and-variational-methods-in-kinetic-theory-may-13-17-2019/

- Bao Quoc Tang discussed with Amit Einav a method for indirect diffusion effects to reaction-diffusion systems, *i.e.* an effective diffusion for the non-diffusive species which is incurred by a combination of diffusion from diffusive species and reversible reactions between the species. This led to the publication [5].
- Amit Einav together with Beatrice Signorello, Tobias Wöhrer and Anton Arnold started the research into the notion of the generalised Fisher Information in defective (but not degenerate) Fokker-Planck equations and its relevance to the sharp convergence to equilibrium rate. They finished [1], in which they study the Goldstein-Taylor equations (a simplified version of a BGK system), and provide a general method to tackle the question of convergence to equilibrium when the relaxation function is not constant in a quantitative way.
- Bao Quoc Tang discussed with Josephine Evans and Dietert Helge on possible connections between indirect diffusion effects and hypocoercivity.
- Julio Backhoff and Junjian Yang started a cooperation on the subject of non-exponential large deviations and Laplace principles for martingales.
- Julio Backhoff and Giacomo Di Gesù had discussions concerning non-standard large deviation principles for Donsker-type approximation schemes.
- Helge Dietert and Esther S. Daus had discussions with André Schlichting and Matthias Erbar about cross diffusion and gradient flows.
- Helge Dietert finished some works on nonlocal cross diffusion and its entropy structure in [4].
- After several talks about hypocoercivity, Helge Dietert started to discuss with Josephine Evans what is the best (spatially-dependend) noise for a linear kinetic model so that the decay rate is as fast as possible.
- Giacomo Di Gesù had discussions with F. Hérau on tunnelling effects for Kinetic Fokker-Planck equations, and with M. Ottobre on long-time behaviour for infinite-dimensional hypoelliptic equations.
- Jo Evans had an ongoing project with Havva Yoldas about applying probabilisitic hypocoercivity methods to run and tumble chemotaxis equations. Jo Evans finished together with Ivan Moyano their preprint [6].
- Esther S. Daus had discussions with S. Jin and L. Liu about mathematical problems related to uncertainty quantification which led to [3], where they studied the nonlinear multi-species Boltzmann equation with random uncertainty coming from the initial data and collision kernel. Moreover, Esther S. Daus discussed with Junjian Yang about mean-field limits and Cucker-Smale flocking models, and with Alexandra Holzinger about the derivation of Shigesda-Kawaskai-Teramoto type cross-diffusion systems from an interacting particle system, which was recently finished in [2].

Follow-up events organized by group members in the spirit of the Junior Trimester Program

- Julio Backhoff together with some colleagues will be organizing a CIRM conference in November 2021 on the subject of mean-field control, games, and the Schrödinger problem.
- Esther S. Daus together with some colleagues will be organizing the Online Hausdorff School *Diffusive Systems* at the Hausdorff School of Advanced Studies in Mathematics (HSM) Bonn, April 12-16, 2021.

Research output: Completed preprints/papers that arose from the trimester program

References

[1] A. Arnold, A. Einav, B. Signorello and T. Wöhrer. *Large time convergence of the non-homogeneous Goldstein-Taylor equation*. To appear in J. Stat. Phys. arXiv:2007.11792.

[2] L. Chen, E. S. Daus, A. Holzinger, A. Jüngel. *Rigorous derivation of population cross-diffusion systems from moderately interacting particle systems.* Submitted for publication, 2020, arXiv:2010.12389.

[3] E. S. Daus, S. Jin, L. Liu: *On the multi-species Boltzmann equation with uncertainty and its stochastic Galerkin approximation.* Submitted for publication, 2019, arXiv:1909.01231.

[4] H. Dietert, A. Moussa. *Persisting entropy structure for nonlocal cross-diffusion systems*. Submitted for publication (2021), arXiv:2101.02893.

[5] A. Einav, J. Morgan, Bao Q. Tang. *Indirect diffusion effect in degenerate reaction-diffusion systems.* SIAM Journal of Mathematical Analysis. Vol. 52 (5) (2020) 4314-4361.

[6] J. Evans, I. Moyano. *Quantitative Rates of Convergence to Equilibrium for the Degenerate Linear Boltzmann Equation on the Torus.* Submitted for publication, 2020, arXiv:1907.12836.

Final report for the Junior Hausdorff Trimester Program "Kinetic Theory", 17-21 June, 2019

In June 2019, we participated in the Junior Hausdorff Trimester Program on Kinetic Theory, and successfully organised a workshop on the topic of "Analytical and Computational Problems for Mixtures and Plasma Dynamics". The organizers are Milana Pavic-Colic, Irene M. Gamba and Liu Liu.
As it is known, modern applications of kinetic equations of collision all type, such as the Boltzmann or Fokker-Planck-Landau equations and models arising in mesoscopic modeling range from rarefied gas mixing and plasma dynamics, to quantum-statistical systems. There are existing and new challenges both in non-linear analysis and numerical approximations, including problems of multi-scale and multi-physics or with uncertainties. We brought together a broad group of researchers with diverse interests to explore and discuss recent progress in these topics.

The speakers of our workshop include both junior and senior distinguished experts in the fields. Here is a full list: Lingbing He, Chiara Saffirio, Berenice Grec, Laurent Boudin, Sergio Simonella, Irene M. Gamba, Jeffrey Haack, Christian Klingenberg, Alexander Bobylev, Liu Liu, Erica Belen De La Canal, Marina Ferreira, Alessia Nota, Shi Jin, Marlies Pinner, Helene Hivert, Helge Dietert, Maxine Herda, Francesco Salvarani, Gernot Heißel, Jang Jin Woo, Anna Szczekutowicz, Raphael Winter, Esther S. Daus, Sonia Akopian and Francois Golse. Though the workshop was held on June 17-21 (except for 20), many of us spent a few days or up to weeks at the Hausdorff Institute and actively participated in frequent academic activities held by this Trimester Program.

We acknowledge the great support of providing this extraordinary platform, both financially and spiritually, by the Hausdorff Institute at University of Bonn for us to organise the workshop, meet new friends, get enlightened through communicating with other researchers, as well as explore potential new collaborations. The generous financial support, especially for junior researchers, really meant a lot. The spacious office space and facilities, high quality coffee and cakes, and the valuable opportunities to meet and interact with both junior and senior scientists from various aspects of kinetic theory, including both analysis and numerical computations, was sincerely and highly appreciated. Our group and visitors were based in several offices in the Hausdorff building, allowing us frequent interactions and a vibrant and exciting discussion atmosphere. The discussions sparked by gatherings and seminars have led to several on-going projects among ourselves, such as conducting error estimates of efficient numerical methods for multi-species kinetic models with uncertainties and practical applications, developing analysis and numerical simulations for plasma or quantum systems, studying interacting particle systems and quantum dynamics, etc.

Last but not least, our group acknowledges deeply the excellent working and accommodation conditions provided by the Hausdorff Research Institute for Mathematics (HIM). We are particularly grateful for the generous funding chances given to organise the workshop and support our visiting, which especially meant significantly for junior researchers on their academia career developments.

Report on the Project *Confined Kinetic Equations*

Junior Trimester Program "Kinetic Theory" (May - August 2019)

M. BRIANT (Univ. Paris), L. CESBRON (É. Polytechnique) & A. TRESCASES (Univ. Toulouse)

1 Scientific goals

Kinetic theory allows the description of a gas (or any physical, biological or abstract substance with similar properties) at a mesoscopic scale, a scale which is intermediate between microscopic and macroscopic. The impacts range from very fundamental questions to practical applications: theoretical bridge between the microscopic and macroscopic theories (reunification of Physics), compromise of intermediate mathematical and numerical complexity, modelling of situations where the fluid approximation is too rough (such as rarefied gas), derivation of new fluid-like models in various contexts (Biology, Social sciences, ...), among others.

In practice, in the vast majority of applications, the gas considered is in interaction with a solid object: sometimes the object of interest is actually the solid itself (eg. in spatial engineering), in other contexts the gas is contained in a box and interacts with the walls. It is therefore essential to study the problem with boundary conditions.

From the mathematical point of view, the situation in bounded domains is extremely complex. In typical cases, the presence of a boundary creates a singularity. The singularity is formed at the boundary, and propagates along the trajectories: therefore, the geometry of the domain plays a fundamental role [5, 6]. In the case of a collisional model, these singularities may be somehow compensated by the regularizing effect of the collisional operator and the solution might be more regular inside the domain than at the boundary. Conversely, if the collisions are neglected, one expects a more singular solution.

The goal of this project is to quantify precisely this singularity by developing a maximal regularity theory with the help of adequate mathematical tools: regularity functional spaces (Sobolev, Hölder, ...) and adapted localisation of the singularity (*kinetic distance*, geometrical description of singular trajectories, etc.).

We started with the most basic problem we could think of: a collisionless equation. Consider then the free transport equation in a bounded domain Ω,

$$\partial_t F + v \cdot \nabla_x F = 0,$$

where $F = F(t, x, v)$ is the density (in space $x \in \Omega \subset \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$) of particles at time $t \geq 0$. This equation is supplemented with the specular boundary conditions (billiard-like reflections),

$$F(t, x, v) = F(t, x, \mathrm{R}_x v), \qquad \text{for } x \in \partial\Omega,$$

where $\mathrm{R}_x = \mathrm{I}_3 - 2\nu(x) \otimes \nu(x)$ is the reflection through the tangential plane to the boundary at the point $x \in \partial\Omega$ (and $\nu(x)$ is the outward normal at point x).

As simple as this equation might look, several fundamental questions are still unanswered such as the maximal regularity of solutions − even in elementary convex domains − due to the conservative nature of the boundary condition [1] as opposed to contracting boundary operators.

We decided to tackle the question of the maximal regularity in two simple geometries to start with: the case of a flat domain (half-space) and the case of the unit ball. Indeed, see [3, 4], the symmetries of the ball sensibly simplify the analysis − for instance due to the fact that the trajectories of the free transport in this setting are 2D − without depriving the problem of its key difficulties. We expected, through the unit ball, to identify the fundamental phenomena which impact the Hölder/Sobolev regularity of solutions. This would allow to tackle the general problem with a much better understanding of the underlying mechanics.

2 Advances

We developed a geometrical approach, based on the unfolding of characteristics, that allows to treat the regularity up to the boundary, grazing boundary excluded, in any convex domain. We complemented this method by an operator algebra approach to treat the singularities at the grazing boundary in flat domains (half-space). Thanks to these methods, we obtained a **maximal regularity theory of the free transport problem in flat domains, in terms of Hölder and (local) Sobolev spaces** [2].

To go beyond flat geometries, we considered the unit ball. We first obtained the regularity far from the grazing (thanks to the unfolding method). We then analyzed that the case of the ball contains two main difficulties: (1) the infinite number of reflections in finite time and (2) the geometry. To understand former we introduced a toy problem that contains difficulty (1) but not difficulty (2): we consider the free transport in a flat domain, and added a force field that attracts particles towards the boundary, so that each particle is attracted to the boundary, then reflected away, attracted to, reflected away, and so on. As for the case without force, we obtained a **maximal regularity theory of the transport problem with attractive force in flat domains, in terms of Hölder and (local) Sobolev spaces** [2].

3 Long-term goals and other projects

The main continuation of this project is to consider the case of the unit ball. As explained before, the main difficulty of the ball lies in the geometrical aspects. In the longer term, we will consider more general geometries, and, finally, add a collision operator (eg. Boltzmann operator). The natural follow-up is then to consider more realistic boundary conditions such as the Maxwell boundary conditions, in which the temperature of the boundary influences the gas.

A. T. would like to also mention an ongoing work with M. BREDEN and M. HERDA that was initiated thanks to fortuitous discussions during the program. The purpose of this work is a link between kinetic theory and cross-diffusion systems that brings new understanding in the modelling.

Acknowlegdement We would like to thank HIM for allowing us to benefit from their beautiful and very stimulative environment. The advances mentioned are a direct consequence of our four-weeks stay working intensively together and interacting with other participants. We would particularly like to thank J. J. L. VELÁZQUEZ who suggested to study a toy model when discussing at HIM. Finally, the program also gave rise to new directions of research in collaboration with other participants.

References

[1] R. Beals and V. Protopopescu. Abstract time-dependent transport equations. *J. Math. Anal. Appl.*, 121(2):370–405, 1987.

[2] M. Briant, Cesbron. L., and A. Trescases. Regularity theory for transport equatins in the half space with specular reflections. *to appear*.

[3] L. Cesbron. Anomalous Diffusion Limit of Kinetic Equations in Spatially Bounded Domains. *Comm. Math. Phys.*, 364(1):233–286, 2018.

[4] L. Cesbron and H. Hutridurga. Diffusion limit for Vlasov-Fokker-Planck Equation in bounded domains. *ArXiv e-prints*, April 2016.

[5] Y. Guo, C. Kim, D. Tonon, and A. Trescases. BV-regularity of the Boltzmann equation in non-convex domains. *Arch. Ration. Mech. Anal.*, 220(3):1045–1093, 2016.

[6] Y. Guo, C. Kim, D. Tonon, and A. Trescases. Regularity of the Boltzmann equation in convex domains. *Invent. Math.*, 207(1):115–290, 2017.

Final Report

Study of a fractional kinetic Fokker-Planck equation

Junior Trimester Program in Kinetic Theory

Group members: Nathalie Ayi, Maxime Breden, Maxime Herda, Hélène Hivert.

Research interests: Fractional Laplacian, Fokker-Planck operator, fractional diffusion, heavy-tailed distribution, (linear) kinetic equations, numerical method, hypocoercivity.

Scientific Activities

The group is grateful for the opportunity to be part of this Junior Trimester Program in Kinetic Theory and warmly acknowledge the HIM for the financial support and the hospitality they benefited during their stay. We also are particularly grateful for the funding provided to organise the workshop below.

Workshop: Qualitative behaviour of kinetic equations and related problems: numerical and theoretical aspects

This workshop focused on qualitative analysis of problems arising in Kinetic theory, both from theoretical and numerical viewpoints. We explored various mathematical directions that are useful to obtain relevant information on the behavior of the solutions, including questions as various as well-posedness and regularity, numerical simulation, long-time or asymptotic behaviour (scale limits, homogenization, asymptotic and long-time preserving schemes) and controllability. The workshop was a very nice event with about 40 participants and 18 talks.

Organisers: Nathalie Ayi, Ariane Trescases, Harsha Hutridurga Ramaiah.

Speakers: Marianne Bessemoulin-Chatard, Maxime Breden, Carlos Castro, Ludovic Cesbron, Anais Crestetto, Helge Dietert, Megan Griffin-Pickering, Jessica Guérand, Frédéric Hérau, Ning Jiang, Laurent Laflèche, Donghyun Lee, Pierre Monmarché, Arnaud Munch, Karel Pravda-Starov, Miguel Rodrigues, Francesco Salvarani, Havva Yoldas.

Research outputs

- **Nathalie Ayi**, **Maxime Herda** and **Hélène Hivert** acknowledge fruitful discussions with Markus Schmidtchen about the analysis of the long time behavior and entropy structure of Fokker-Planck type equations.

- **Nathalie Ayi** acknowledges fruitful discussions with Raphael Winter and Juan Velázquez about discrete velocity models and the derivation of mesoscopic models starting from the microscopic ones.

- **Maxime Breden** and **Maxime Herda** initiated discussions with Ariane Trescases on the derivation of nonlinear parabolic systems of PDEs with cross-diffusion from kinetic systems of PDEs. Together, they proposed a new kinetic description of particle dynamics with velocities depending on the particle density. From this mesoscopic model, they showed that in the appropriate diffusive limit, velocity averages of the distribution function converge to a solution of the macroscopic model. This work is in progress and will be concluded in a forthcoming preprint [BHT21].

- **Hélène Hivert** met Havva Yoldaş during the Junior Trimester Programm "Kinetic theory" and she proposed her to continue to work together. Havva Yoldaş has been proposed a postdoc position at ICJ, Lyon (France). They worked on asymptotic preserving schemes, for selection-mutation equations arising in population dynamics. In a regime of long time and small mutations, the solution of this equation converges to the viscosity solution of a constrained Hamilton-Jacobi equation. The scheme they proposed enjoys stability properties in the transition to the asymptotic regime. In particular, the viscosity solution of the limit problem is well approximated by the asymptotic scheme. They discussed the construction of the scheme and proved of its properties.

- **Nathalie Ayi**, **Maxime Herda**, **Hélène Hivert** and Isabelle Tristani work on the collaborative research project that was initiated for the Junior Trimester Program in Kinetic Theory at the HIM. It leads to two papers.
  In the first one [AHHT20], we are interested in the large time behavior of linear kinetic equations with heavy-tailed local equilibria. Our main contribution concerns the kinetic Lévy-Fokker- Planck equation, for which we adapt hypocoercivity techniques in order to show that solutions converge exponentially fast to the global equilibrium. Compared to the classical kinetic Fokker-Planck equation, the issues here concern the lack of symmetry of the non-local Lévy-Fokker- Planck operator and the understanding of its regularization properties. As a complementary related result, we also treat the case of the heavy-tailed BGK equation. This paper has been published in *Comptes Rendus. Mathématique.*
  In the second one [AHHT21], we introduce and analyse numerical schemes for the homogeneous and the kinetic Léevy-Fokker-Planck equation. The discretizations are designed to preserve the main features of the continuous model such as conservation of mass, heavy-tailed equilibrium and (hypo)coercivity properties. We perform a thorough analysis of the numerical scheme and show exponential stability. Along the way, we introduce new tools of discrete functional analysis, such as discrete nonlocal Poincaré and interpolation inequalities adapted to fractional diffusion. Our theoretical findings are illustrated and complemented with numerical simulations.

- **Nathalie Ayi** and **Hélène Hivert** initiated discussions about radiative transfer type equations with heavy-tailed equilibrium. In a work in progress, they consider anomalous diffusion limits and numerical considerations associated with it.

References

[AHHT20] N. Ayi, M. Herda, H. Hivert, and I. Tristani. A note on hypocoercivity for kinetic equations with heavy-tailed equilibrium. *Comptes Rendus. Mathématique*, 358(3):333–340, 2020.

[AHHT21] N. Ayi, M. Herda, H. Hivert, and I. Tristani. On a structure-preserving numerical method for fractional Fokker-Planck equations. Submitted preprint, July 2021.

[BHT21] M. Breden, M. Herda, and A. Trescases. On the derivation of nonlinear parabolic systems as diffusive limits of systems of kinetic equations with heterogeneous velocities. Preprint in preparation, 2021.